Period 3, May 7, 2025

(c) The grain in the silo is spoiling at a rate modeled by $w(t) = 32 \cdot \sqrt{\sin\left(\frac{\pi t}{74}\right)}$, where $w(t)$ is measured in cubic feet per minute for $0 \le t \le 8$ minutes. Using the result from part (b), approximate the amount of unspoiled grain remaining in the silo at time $t = 8$.

$$\int_0^8 32\sqrt{\sin\frac{\pi T}{74}}\,dT = 99.051$$

Part b = $160.8\ FT^3$

$160.8 - 99.051 = 61.749\ FT^3$

(d) Based on the model in part (c), is the amount of unspoiled grain in the silo increasing or decreasing at time $t = 6$? Show the work that leads to your answer.

Rate of Spoil at $T = 6$

Rate of Spoil $w(6) = 32 \cdot \sqrt{\sin\frac{6\pi}{74}} = 16.063$

$g(6) = 18.3$ amount added

$g(6) - w(6) > 0$ unspoiled grain increasing

$g(6) - w(6) < 0$ unspoiled grain decreasing

Rate of grain added

$18.3 - 16.063 = +$

$= 2.237$ Rate of unspoiled grain

2. A snail is traveling along a straight path. The snail's velocity can be modeled by $v(t) = 1.4\ln\left(1+t^2\right)$ inches per minute for $0 \le t \le 15$ minutes.

$a(T) = V'(T) = 1.4 \cdot \frac{1}{1+T^2} \cdot 2T = \frac{2.8T}{1+T^2}$

(a) Find the acceleration of the snail at time $t = 5$ minutes.

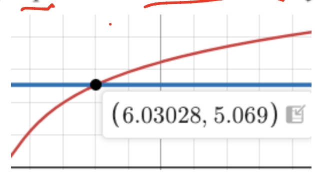
$\frac{d}{dT}(v(T)) = a(T)$

$a(5) = \frac{2.8(5)}{1+(5)^2} = 0.538 \text{ in/min}^2$

(b) What is the displacement of the snail over the interval $0 \le t \le 15$ minutes?

$X(T) = \int_0^{15} v(T)\,dT = \int_0^{15} 1.4\ln(1+T^2)\,dT = 76.043 \text{ in}$

(c) At what time t, $0 \le t \le 15$, is the snail's instantaneous velocity equal to its average velocity over the interval $0 \le t \le 15$?



Average velocity $= \frac{76.043}{15} = 5.069$

$V(6.03028) = 5.069$

$T = 6.03028$

(d) An ant arrives at the snail's starting position at time $t = 12$ minutes and follows the snail's path. During the interval $12 \le t \le 15$ minutes, the ant travels in the same direction as the snail with a constant acceleration of 2 inches per minute per minute. The ant catches up to the snail at time $t = 15$ minutes. The ant's velocity at time $t = 12$ is B inches per minute. Find the value of B.

ant

$a(T) = 2$

Snail $T=0$ 76.043 in $T=15$

anT $12=T$ 76.043 $15=T$

$V(T) = \int a(T)\,dT = \int 2\,dT$

$V(T) = 2T + C$

$\int_{12}^{15} (2T+C)\,dT = 76.043$

$T^2 + CT + C_2 \Big]_{12}^{15} = 76.043$

$15^2 + 15C - (12^2 + 12C) = 225 + 15C - 144 - 12C = 81 + 3C = 76.043$

$c = -1.652309$

$V_{anT} = 2T - 1.652309$

$V(12) = 2(12) - 1.652309 = 22.348 \text{ in/min}$

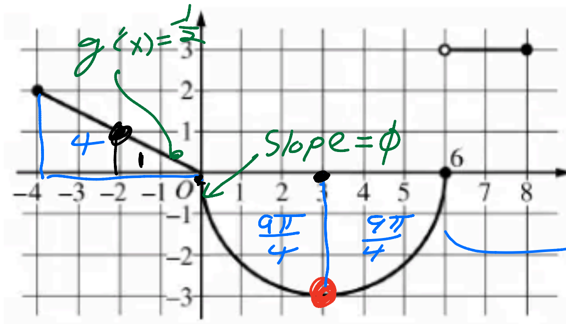
$(x-h)^2+(y-k)^2=r^2$

$(x-3)^2+(y-0)^2=3^2$

$x^2-6x+9+y^2=9$

$2x-6+2y\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{6-2x}{2y}=\frac{0}{0}=\phi \quad x=0,\ y=0$



Graph of g

Semicircle $r=2$

$Area=\frac{\pi r^2}{2}=\frac{9\pi}{2}$

3. The function g is defined on the closed interval $[-4, 8]$. The graph of g consists of two linear pieces and a semicircle, as shown in the figure above. Let f be the function defined by $f(x) = 3x + \int_0^x g(t)\,dt$.

(a) Find $f(7)$ and $f'(7)$.

$F'(x)=3+g(T)=$ always $+$ — always increas from $x=-4$ to $x=3$

(b) Find the value of x in the closed interval $[-4, 3]$ at which f attains its maximum value. Justify your answer.

$F'(x)=0$ or ϕ

$F'(x)=3+g(x)$ — always increasing, Highest Point at End

$F'(x)$ is not $=\phi$

$0=3+g(x)$

$-3=g(x)$

$-3=g(3)$

x	y
-4	-16
3	$9-\frac{9\pi}{4}=+$
3	

$F(-4)=3(-4)+\int_0^{-4} g(T)\,dT=-12-4$

$F(3)=3(3)+\int_0^3 g(T)\,dT=9-\frac{9\pi}{4}$

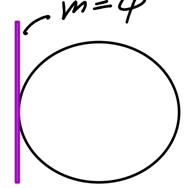
(c) For each of $\lim_{x\to 0^-} g'(x)$ and $\lim_{x\to 0^+} g'(x)$, find the value or state that it does not exist.

$g'(x)=$ Slope of $g(x)$

$\lim_{x\to 0^-} g'(x)=-\frac{1}{2}$

$\lim_{x\to 0^+} g'(x)=\phi$

$m=\phi$



(d) Find $\lim_{x\to -2} \frac{f(x)+7}{e^{3x+6}-1}=\frac{-7+7}{e^0-1}=\frac{0}{0}$

$\lim_{x\to -2}\frac{F(x)+7}{e^{3x+6}-1}=\lim_{x\to -2}\frac{3+g(T)}{e^{3x+6}\cdot 3}$

$F(-2)=\int_0^{-2} g(x)\,dx+3(-2)=-1-6=-7$

$g(-2)=1$

$\frac{3+g(-2)}{e^{3(-2)+6}\cdot 3}=\frac{3+1}{1\cdot 3}=\frac{4}{3}$

e^0

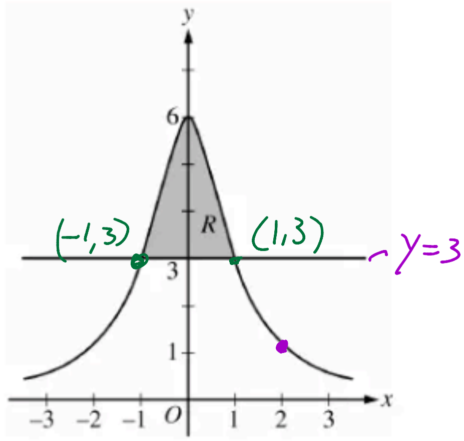
$$\frac{6}{1+x^2}=3$$

$$6=3(1+x^2)$$

$$6=3+3x^2 \rightarrow 1=x^2$$

$$-3 \quad -3 \qquad \pm 1 = x$$

$$\frac{3}{3}=\frac{\cancel{3}x^2}{\cancel{3}}$$



4. Let f be the function defined by $f(x) = \dfrac{6}{1+x^2}$. Let R be the shaded region bounded by the graph of f and the horizontal line $y = 3$, as shown in the figure above.

(a) Find the area of R.

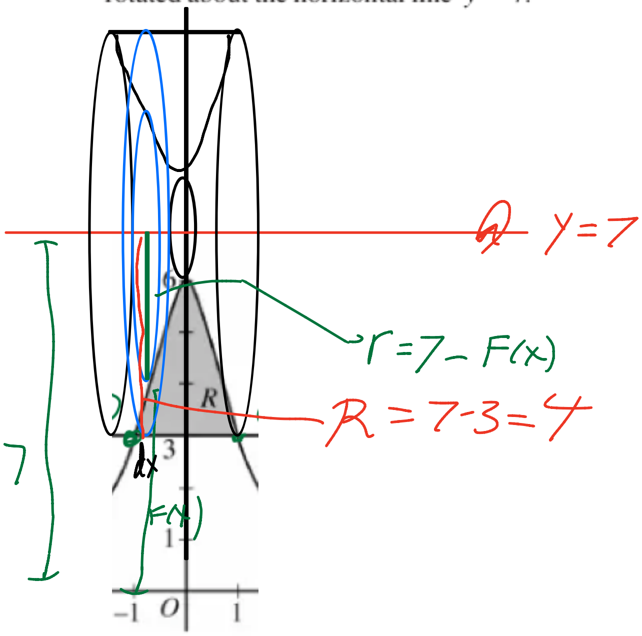
$$\int_{-1}^{1}\left(\frac{6}{1+x^2}-3\right)dx = 2\int_{0}^{1}\left(\frac{6}{1+x^2}-3\right)dx =$$

$$6\arctan x - 3x \Big[_{-1}^{1} = 6\arctan 1 - 3(1) - \left[6\arctan -1 - 3(-1)\right]$$

$$\frac{6\pi}{4} - 3 + \frac{6\pi}{4} - 3 \qquad \frac{-\pi}{4} \quad +3$$

$$\frac{12\pi}{4} - 6 = 3\pi - 6$$

(b) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 7$.



$$\pi\int_{-1}^{1}(R^2-r^2)\,dx$$

$$\pi\int_{-1}^{1}\left[(4)^2-(7-F(x))^2\right]dx$$

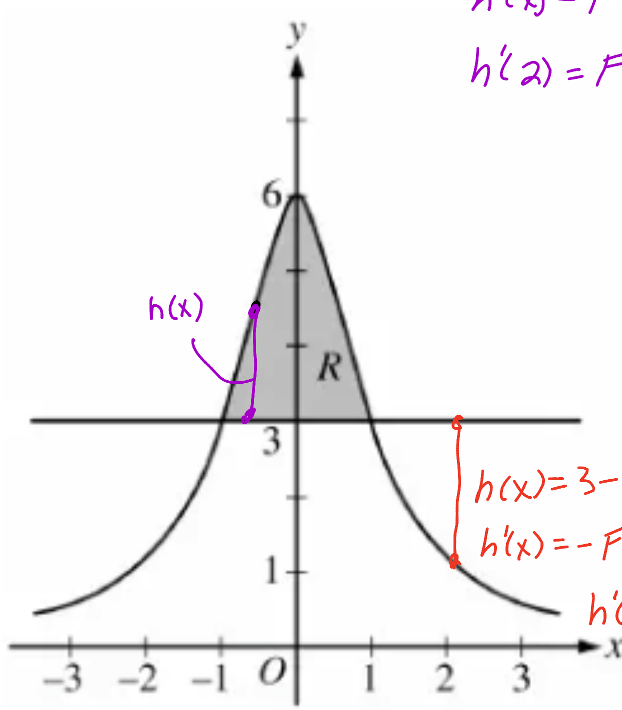
$$\pi\int_{-1}^{1}\left[16-\left(7-\frac{6}{1+x^2}\right)^2\right]dx$$

(c) Let $h(x)$ be the vertical distance between the point $(x, f(x))$ and the horizontal line $y = 3$. Find the rate of change of $h(x)$ with respect to x at $x = 2$.

$h(x) = F(x) - 3 \quad [-1, 1]$

$h'(x) = F'(x) + 0$

$h'(2) = F'(2)$



$F(x) = \frac{6}{1+x^2} = 6(1+x^2)^{-1}$

$F'(x) = 6 \cdot -1 (1+x^2)^{-2} \cdot 2x$

$F'(x) = \frac{-12x}{(1+x^2)^2}$

$F'(2) = \frac{-12(2)}{(1+2^2)^2} = \frac{-24}{25}$

$h(x) = 3 - F(x)$

$h'(x) = -F'(x)$

$h'(2) = -F'(2) = -\left(\frac{-24}{25}\right) = \frac{24}{25}$

5. During a chemical reaction, the function $y = f(t)$ models the amount of a substance present, in grams, at time t seconds. At the start of the reaction ($t = 0$), there are 10 grams of the substance present. The function $y = f(t)$ satisfies the differential equation $\frac{dy}{dt} = -0.02y^2$.

y = Substance Present

$T = 0 \quad y = 10$

(a) Use the line tangent to the graph of $y = f(t)$ at $t = 0$ to approximate the amount of the substance remaining at time $t = 2$ seconds.

$y = 10$

Slope $= \frac{dy}{dT} = -0.02(10)^2 \approx -2 \quad$ Point $(0, 10)$

$y - 10 \approx -2(T - 0)$

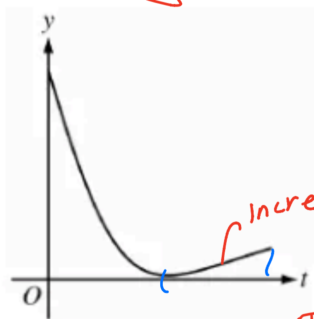
$y \approx -2T + 10 \qquad y = -2(2) + 10 \approx 6$ grams

(b) Using the given differential equation, ~~determine whether~~ the graph of f could resemble the following graph. Give a reason for your answer.

$$\frac{dy}{dT} = -0,02y^2$$

$\frac{dy}{dT}$ always $-$

Slope is always $-$
graph is decreasing



Bad

(c) Find an expression for $y = f(t)$ by solving the differential equation $\frac{dy}{dt} = -0.02y^2$ with the initial condition $f(0) = 10$.

$$\frac{dT}{y^2}\cdot\frac{dy}{\cancel{dT}} = \frac{-0.02\cancel{y^2}\cdot dT}{\cancel{y^2}}$$

$$\int \frac{1}{y^2}\,dy = \int -0.02\,dT$$

$$\int y^{-2}\,dy = \int -0.02\,dT$$

$$\frac{1}{-1}y^{-2+1=-1} = -0.02T + C$$

$$\frac{-1}{y} = -0.02T + C \qquad (0, 10)$$

$$\frac{-1}{10} = \cancel{-0.02(0)} + C$$

$$\cancel{-y\cdot}\ \frac{-1}{y} = \left(-0.02T - \frac{1}{10}\right)\cdot -y$$

$$\frac{1}{0.02T + \frac{1}{10}} = \frac{1 = \left(-0.02T - \frac{1}{10}\right)\cdot + y}{-\left(-0.02T - \frac{1}{10}\right) + \cancel{\left(-0.02T - \frac{1}{10}\right)}}$$

(d) Determine whether the amount of the substance is changing at an increasing or a decreasing rate. Explain your reasoning.

$\frac{dy}{dt} = -0.02y^2$

$\frac{d^2y}{dt^2} = -0.02 \cdot 2y' \cdot \frac{dy}{dt}$

$-0.04y' \cdot (-0.02y^2)$

$0.008y^3$

$\frac{d^2y}{dt^2}$ = + and − decreasing

increasing

y = amount of substance

y = positive

6. Consider the curve given by the equation $2(x-y) = 3 + \cos y$. For all points on the curve, $\frac{2}{3} \leq \frac{dy}{dx} \leq 2$.

(a) Show that $\frac{dy}{dx} = \frac{2}{2 - \sin y}$.

$\frac{d}{dx}[2x - 2y = 3 + \cos y]$

$2 - 2\frac{dy}{dx} = 0 - \sin y \frac{dy}{dx}$

$+2\frac{dy}{dx} \qquad +2\frac{dy}{dx}$

$2 = 2\frac{dy}{dx} - \sin y \frac{dy}{dx} = \frac{dy}{dx}(2 - \sin y)$

$\frac{2}{2 - \sin y} = \frac{\cancel{(2 - \sin y)}\frac{dy}{dx}}{\cancel{2 - \sin y}}$

(b) For $-\frac{\pi}{2} < y < \frac{\pi}{2}$, there is a point P on the curve through which the line tangent to the curve has slope 1. Find the coordinates of the point P.

$$\frac{dy}{dx} = 1 = \frac{2}{2-\sin y} \cdot (2-\sin y)$$

$(2-\sin y)$

$$2 - \sin y = 2$$

$-2 \qquad -2$

$$-\sin y = 0$$

$$\sin y = 0$$

$$y = 0$$

$$2(x-0) = 3 + \cos 0$$

$$2x = 3+1$$

$$2x = 4$$

$$x = 2$$

$(2, 0)$

(c) Determine the concavity of the curve at points for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Give a reason for your answer.

$$\frac{dy}{dx} = \frac{2}{2-\sin y} \qquad \frac{d^2y}{dx^2} \qquad \cos y = +$$

$$\frac{dy}{dx} = 2(2-\sin y)^{-1}$$

$$\frac{d^2y}{dx^2} = 2 \cdot -1(2-\sin y)^{-2} \cdot -\cos y \cdot \frac{dy}{dx}$$

$$\frac{2\cos y}{\underbrace{(2-\sin y)^2}_{+}} \cdot \frac{2}{2-\sin y} = + \qquad = \frac{d^2y}{dx^2} = + \quad \text{concave up}$$

$[1, -1]$

(d) Let $y = f(x)$ be a function, defined implicitly by $2(x - y) = 3 + \cos y$, that is continuous on the closed interval $[2, 2.1]$ and differentiable on the open interval $(2, 2.1)$. Use the Mean Value Theorem on the interval $[2, 2.1]$ to show that $\frac{1}{15} \leq f(2.1) - f(2) \leq \frac{1}{5}$.

$y = F(2) = 0$ $\quad \frac{1}{15} < F(2.1) - 0 < \frac{1}{5}$

$y = F(2.1)$

$\frac{dy}{dx} = \frac{2}{2 - \sin y}$

$\frac{dy}{dx} = \frac{F(x_1) - F(x_2)}{x_1 - x_2}$

$F'(x) = \frac{F(2.1) - F(2)}{0.1}$

$\frac{2}{3} < \frac{dy}{dx} < 2$

$0.1 \cdot \frac{2}{3} < 0.1 \cdot \frac{F(2.1) - F(2)}{0.1} < 2(0.1)$

$\frac{.2}{3} < F(2.1) - F(2) < .2$

$\frac{1}{15} < F(2.1) - F(2) < \frac{1}{5}$